TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 4 2018

Section 1 Calculator Free

Integration and Applications of Integration

STUDENT'S NAME SOLUTIONS

DATE: Friday 20 July **TIME:** 36 minutes **MARKS:** 36

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Determine $\int x\sqrt{x-3}\,dx$ let $u = x-3$ $x = u+3$

$= \int (u+3)u^{\frac{1}{2}}\,du$ $\frac{du}{dx} = 1$

$= \int u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\,du$ $du = dx$

$= \frac{2u^{\frac{5}{2}}}{5} + 3 \times \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2(x-3)^{\frac{5}{2}}}{5} + 2(x-3)^{\frac{3}{2}} + c$

2. (9 marks)

(a) $\int e^x \sin(e^{x+3})dx$ let $u = e^{x+3}$ [4]

$u = e^3 e^x$

$\frac{u}{e^3} = e^x$

$\frac{du}{dx} = e^{x+3}$

$\frac{du}{e^{x+3}} = dx$

$\frac{du}{u} = dx$

$= \int \frac{u}{e^3} \sin u \frac{du}{u}$

$= \frac{1}{e^3}\int \sin u \, du$

$= -\frac{\cos u}{e^3} + c$

$= -\frac{\cos(e^{x+3})}{e^3} + c$

(b) $\int_2^e \frac{1}{x\ln\sqrt{x}}dx$ let $u = \ln\sqrt{x}$ [5]

$u = \frac{1}{2}\ln x$

$\frac{du}{dx} = \frac{1}{2x}$

$2x\,du = dx$

$x=e \quad u = \frac{1}{2}\ln e = \frac{1}{2}$

$x=2 \quad u = \frac{1}{2}\ln 2$

$= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}} \frac{1}{\cancel{x}u} 2\cancel{x}\,du$

$= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}} \frac{2}{u}\,du$

$= \left[2\ln u\right]_{\frac{1}{2}\ln 2}^{\frac{1}{2}}$

$= 2\ln 2^{-1} - 2\ln\left(\frac{1}{2}\ln 2\right)$

$= -2\ln 2 - 2\ln(\ln\sqrt{2})$

$= -2\ln\left(\frac{2}{\ln\sqrt{2}}\right)$

$= -2\ln(\ln 2)$

ANY CORRECT FORM

3. (9 marks)

(a) $\int \sin^3 t \cos^2 t \, dt$ let $x = \cos t$ [5]

$$\frac{dx}{dt} = -\sin t$$

$$\frac{-dx}{\sin t} = dt$$

$$= \int \sin^3 t \; x^2 \frac{-dx}{\sin t}$$

$$= -\int \sin^2 t \; x^2 \, dx$$

$$= -\int (1 - \cos^2 t) \, x^2 \, dx$$

$$= -\int (1 - x^2) \, x^2 \, dx$$

$$= -\int x^2 - x^4 \, dx$$

$$= -\frac{x^3}{3} + \frac{x^5}{5} + c$$

$$= -\frac{\cos^3 t}{3} + \frac{\cos^5 t}{5} + c$$

(b) $\int \frac{8x-7}{2x-3} dx$ [4]

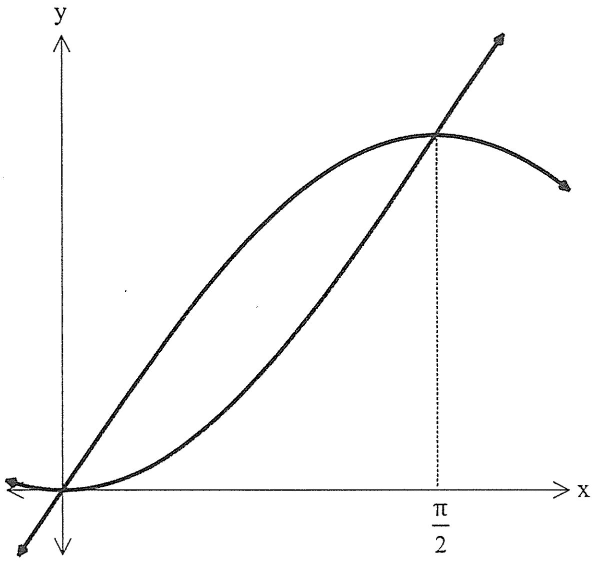
$$\begin{array}{r|l} & 4 \\ 2x-3 & 8x-7 \\ & -(8x-12) \\ \hline & 5 \end{array}$$

$$= \int 4 + \frac{5}{2x-3} \, dx$$

$$= \int 4 \, dx + \frac{5}{2} \int \frac{2}{2x-3} \, dx$$

$$= 4x + \frac{5}{2} \ln|2x-3| + c$$

4. (7 marks)



Shown above are the functions $y = \sin x$ and $y = 1 - \cos x$. The area enclosed between the two graphs is rotated about the x-axis. Determine the exact volume of the solid created.

$$V_x = \pi \int y^2 \, dx$$

$$\text{VOLUME} = \pi \int_0^{\frac{\pi}{2}} \sin^2 x - (1 - \cos x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x - (1 - 2\cos x + \cos^2 x) \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x - \cos^2 x - 1 + 2\cos x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} -\cos 2x - 1 + 2\cos x \, dx$$

$$= \pi \left[-\frac{\sin 2x}{2} - x + 2\sin x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\left(0 - \frac{\pi}{2} + 2\right) - (0 - 0 + 0) \right]$$

$$= 2\pi - \frac{\pi^2}{2}$$

5. (7 marks)

(a) Determine $\int \frac{4x+2}{x^2+x-2}\,dx$ [2]

$= 2\int \frac{2x+1}{x^2+x-2}\,dx$

$= 2\ln|x^2+x-2| + c$

(b) Determine $\int \frac{2x+10}{x^2+x-2}\,dx$ [5]

$= \int \frac{-2}{x+2} + \frac{4}{x-1}\,dx$

$= -2\ln|x+2| + 4\ln|x-1| + c$

$\frac{2x+10}{(x-2)(x+1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$\frac{2x+10}{(x-2)(x+1)} = \frac{Ax - A + Bx + 2B}{(x+2)(x-1)}$

$\therefore A + B = 2$

$-\ (-A + 2B = 10\ )$

$3B = 12$

$B = 4$

$A = -2$

TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 2 2018

Section 2 Calculator Assumed
Integration and Applications of Integration

STUDENT'S NAME ______________________________

DATE: Friday 20 July **TIME:** 14 minutes **MARKS:** 14

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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6. (6 marks)

The function $y = f(x)$ is a continuous curve in the first quadrant. Some values are shown in the table below.

x	0	1	2	3	4	5
$f(x)$	8.9	11.7	14.3	16.6	18.6	20.4

(a) Determine an approximation for the area between the curve and the x-axis for $1 \le x \le 4$ by summing the areas of trapeziums. [4]

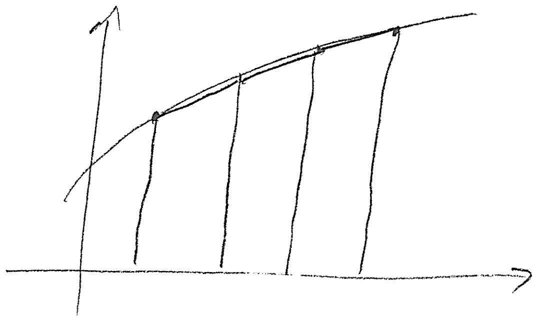
$$AREA = \left(\frac{A+B}{2}\right) ht$$

$$AREA = \left(\frac{14.3+11.7}{2}\right)1 + \left(\frac{16.6+14.3}{2}\right)1 + \left(\frac{18.6+16.6}{2}\right)1$$

$$= 13 + 15.45 + 17.6$$

$$= 46.05$$

(b) Is the estimation in (a) less than or greater than the exact area? Justify your answer. [2]



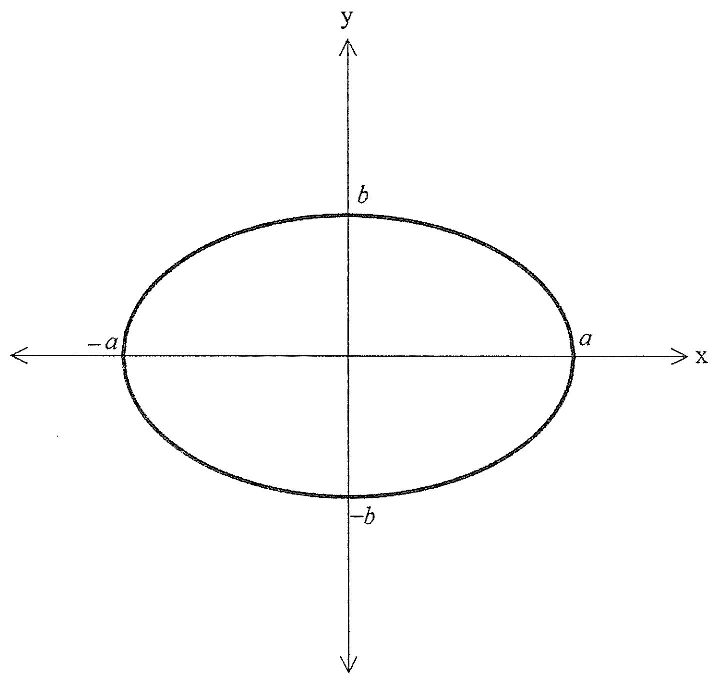
GRADIENT DECREASING ∴ TOP OF TRAPEZIUM BELOW CURVE

LESS THAN EXACT AREA

7. (8 marks)

The ellipse drawn below is centred at the origin and has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or

$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$ for $a,b>0$



(a) Show the area of the ellipse is given by $\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$ [2]

TOTAL AREA = 4 × AREA IN FIRST QUADRANT

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

(b) Using the substitution $x = a\sin\vartheta$, show the exact area of the ellipse is $ab\pi$. [6]

$$\text{AREA} = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\sqrt{1-\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \frac{4a^2b}{a}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 4ab\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 4ab\left[\left(0 + \frac{\pi}{4}\right) - (0+0)\right]$$

$$= ab\pi$$

$x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta\, d\theta$

$x = a \quad a = a\sin\theta$

$1 = \sin\theta$

$\frac{\pi}{2} = \theta$

$x = 0 \quad 0 = a\sin\theta$

$0 = \sin\theta$

$0 = \theta$